Analyzing Structural Gaps in Mathematical Argumentation: A Toulmin-Based Study on Graph Theory

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Abstract: This research project sought to explain the format and quality of students' mathematical argumentation in graph theory by examining how students built and defended arguments using the Toulmin model. Although research on mathematical argumentation has been extensive, studies that explicitly examine the structure of students' argumentation in graph theory are still very limited, especially in the context of Discrete Mathematics courses in Indonesian higher education. The qualitative descriptive design has been used to investigate students' written responses to graph theory problems in a Discrete Mathematics course. The sample consisted of 22 undergraduate students from the Mathematics Education Study Program at Universitas Singaperbangsa Karawang, selected purposively and classified into high-, medium-, and low-ability groups. Inductive and deductive analysis methods were applied to the data to identify patterns in the reasoning and to assess whether the students' arguments were complete and logically consistent. Data analysis was conducted by combining inductive and deductive approaches supported by a Toulmin model-based coding framework to identify the structure and completeness of arguments, particularly the presence and thickness of claims, data, and warrants, and to compare patterns across levels of ability. The findings showed clear differences in mathematical argumentation across ability levels. Students with high ability presented more coherent arguments with correct and justified claims and logical warrants, whereas medium- and low-ability students produced incomplete or no arguments. The results of this study suggest that ways to enhance the reasoning and argumentation of mathematics instruction, especially by using tasks that encourage justification and conceptual learning in discrete mathematics, need to be reinforced.

Keywords: mathematical argumentation, reasoning, Toulmin's model, graph theory, discrete mathematics, mathematics education.

▪ **INTRODUCTION**

As a component of reasoning and proof, argumentation has become progressively more important in mathematics education (Guerrier et al., 2012; Francisco, 2022) . In addition to problem-solving or formula application, students are expected to defend their reasoning, assess alternative viewpoints, and construct arguments to explain mathematical phenomena. These are the core skills towards the acquisition of profound conceptual and reflective mathematical thinking (Nieto et al., 2023). Nonetheless, nurturing such an argument in classroom practice is a complex task, particularly when students are required to correlate procedures with the underlying theory in abstract subjects such as graph theory.

Mathematics learning activities involving argumentation have been shown to improve not only students' reasoning and critical thinking but also their communication and social interaction skills. Rapanta & Felton (2022) note that argumentative dialogue can both develop cognitively and socially as students get to share and assess various perspectives. Argumentation is also essential for transforming personal views, as it

enables learners to develop and redefine their thoughts in a critical manner. Cress & Kimmerle (2023) also describe how argumentation promotes change in knowledge, reorganizing it into more intricate and significant connections.

In mathematics, argumentation is strictly connected with reasoning and proof. Reasoning and thought processes are usually related to argumentation. According to Soekisno (2015) Argumentation provides a basis for presenting critical reasoning, backed by pertinent data and theory, to logically explain mathematical problems. On the same note, mathematical argumentation as defined by Indrawatiningsih et al. (2020) refers to an activity that is consistent with the process of proving, which is employed to confirm the truth of statements in problem-solving. Validity can be used to test the quality of an argument: a valid argument is one in which the premises make logical sense and yield a true conclusion. Quality arguments are defined by the statements that have evidence, logicality between the information and the statements, and theoretical support (Dawson & Venville, 2009).

Various models have been proposed to analyze students' argumentation, including those by Mcneill (2008) and Toulmin (2003) the framework created by McNeill is based on three parts: claim, evidence, and reasoning, which help motivate students to develop their positions, provide supporting information, and explain their conclusions. The Toulmin model of argumentation, in turn, distinguishes six points: data, claim, warrant, backing, qualifier, and rebuttal. Toulmin's Argumentation Model was chosen as the analytical framework because it provides a comprehensive argument structure that not only covers claims and evidence but also allows for the identification of logical relationships (warrants), theoretical bases (backing), levels of certainty (qualifiers), and potential exceptions or objections (rebuttals). This structure is particularly well-suited to the context of mathematics/graph theory, where student arguments often involve definitions, axioms, theorems, and examples of graphs as data, as well as claims about the properties of graphs as conclusions. Nevertheless, in recent research, it was found that the mathematical argumentation skills of students are still poor (Aberdein, 2005; Mukuka & Tatira, 2025).

This model has proven to be flexible and can be applied both for analyzing arguments and for designing learning activities that strengthen students' reasoning, justification, and mathematical argumentation skills (Badjeber et al., 2024). Indrawatiningsih et al. (2020) established that approximately 69 percent of students were unable to demonstrate a proper level of argumentation due to conceptual and procedural misconceptions. Similarly, Arizmendi et al. (2024) found mathematics students to have difficulty in developing deductive reasoning and supporting generalizations, whereas Kartika et al. (2024) discovered the poorly developed area of evidence organization, argument validity testing, and logical consistency maintenance in proofs. These findings indicate that mathematical argumentation skills remain a challenge in higher education, necessitating a learning context that fosters formal reasoning. Several studies show that graph theory provides a problem-solving environment that encourages students to construct claims, provide justifications, and produce mathematical proofs (González et al., 2021). In line with González et al. (2024), who shows that proving graph theory material requires students to construct complete, structured, and logically consistent arguments, the context of graph theory can strengthen their mathematical argumentation skills.

Graph theory is one of the mathematical areas with high prospects for this development. Graphs are very effective not only in modeling real-life systems like transportation or social networks, but also in developing the reasoning and argumentation skills of students (Ferrarello et al. 2022). Problems related to graphs will help students make rational arguments, justify their answers, and relate procedural practice to theoretical concepts. Sevcikova et al. (2023) state that graph theory learning encourages students to understand proof structures and develop mathematical arguments through the analysis of relationships between concepts, while Dafik et al. (2020) emphasize that graph-based problem-solving activities are effective in fostering higher-order thinking skills such as analysis, justification, and logical reasoning. Most recent studies have also highlighted the significance of discrete mathematics, such as combinatorics, graph theory, and recursions, as a lucrative setting in which to reason, construct arguments, and make proofs (Sandefur et al., 2022). In this sphere, students are to build, defend, and critique Euler and Hamilton circuits based on conceptual knowledge, not on procedures.

Although there is potential, there is a dearth of empirical research on how students demonstrate their ability to argue within the framework of graph theory. The majority of the available studies emphasize the general reasoning or proof building as opposed to the particular format of the mathematical arguments presented by students of various abilities (Nasution et al., 2020; González et al., 2021; Maliana & Fuady, 2024; Sevcikova et al., 2023). This gap evidences the need for a more thorough elaboration of the process by which students construct arguments, create claims, introduce facts and warrants, and connect their arguments to theoretical support, as they work on graph theory problems. In line with this, this paper investigates the analytical structure of students' mathematical arguments in graph theory by analyzing not only the patterns produced but also the logical components and theoretical justifications underlying these arguments. Rather than simply describing the quality of students' responses, this study aims to analyze how students construct, justify, and connect their claims, data, and support in accordance with Toulmin's Argumentation Model. Through this analytical lens, this research seeks to uncover the types of reasoning students use, the robustness of their argument structures, and the points at which their arguments break down or become incomplete. The results of this study are expected to broaden understanding of how mathematical arguments develop in discrete mathematics, especially in graph theory, and to offer practical implications for designing better teaching practices to develop reasoning, proof, and argumentation in mathematics education. The research questions in this study are

1. How do students construct mathematical arguments in solving graph theory problems based on Toulmin's framework?
2. What reasoning processes and structural weaknesses appear in students' arguments, and how do these affect the quality of their mathematical arguments?

▪ **METHOD**
Participants

This study involved twenty-two students from the Mathematics Education Study Program at Singaperbangsa University Karawang, Indonesia, who were taking the Discrete Mathematics course in the odd semester of 2025/2026. Participants were selected purposively, as all students had systematically studied graph theory material and had
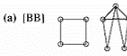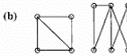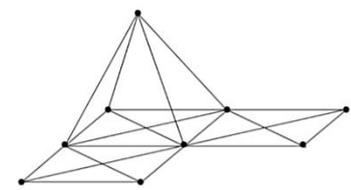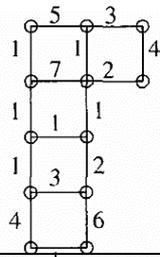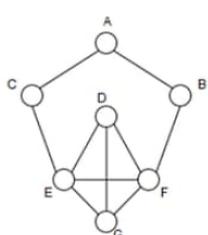
taken the Discrete Mathematics Midterm Exam, which served as the primary data source for this study. Although the sample came from a single institution and was small, this selection was deemed appropriate for the research objective, which prioritized a qualitative deepening of the argumentation structure rather than statistical generalization. Furthermore, although specific features of mathematical argumentation vary across contexts, previous research has identified structural elements that emerge across them. For example, secondary school instruction has shown that students can construct mathematical arguments across a variety of tasks (Nusantara et al., 2025). Research on classroom practices has also observed a recurring three-step argumentation pattern (claim-argument-feedback) across classrooms, indicating consistent structural features (Bredow & Knipping, 2025). Furthermore, a framework for analyzing teachers' attention to argumentation suggests that these structural elements can be identified consistently across contexts (Ayalon, M., 2024). In this section, the argumentation structures identified in this study's graph-theoretic data can provide broader insights into how students formulate mathematical arguments.

Research Design

This research is a qualitative study with a quantitative approach, in which students' written responses were analyzed qualitatively, coded deductively using the Toulmin framework, and then converted into numerical scores as part of the qualitative data quantification process.

Instruments

The instrument used was a Discrete Mathematics midterm exam containing five groups of questions.

Research Instrument	
1. For each pair of graphs shown, determine whether the graph on the left is a subgraph of the graph on the right. If so, label the vertices of the subgraph, then use the same symbols to indicate the corresponding vertices in the graph on the right. If not, explain why the graph is not a subgraph.	2. Is the following graph planar? Give your reasoning based on Kuratowski's theorem!
3. The solution to the Chinese Postman Problem below is....	4. Based on the graph in the figure below, determine and explain:

	a.	What are Eulerian and Hamiltonian circuits?
	b.	Is the graph Eulerian, semi-Eulerian, or neither? Explain your reasoning.
	c.	Draw the Eulerian and Hamiltonian circuits/paths of the graph, if they exist.
5.	A class has 6 task groups, each of which holds one discussion per month. The six task groups and their respective members are: K_1 = {Anisa, Asya, Tarizka} K_2 = {Asya, Hasan, Tommy} K_3 = {Anisa, Tommy, Tarizka} K_4 = {Hasan, Tommy, Tarizka} K_5 = {Anisa, Asya} K_6 = {Asya, Tommy, Tarizka}. How many different discussion times must be planned so that no task group members are scheduled for discussions at the same time? Draw a graph representing this problem, then explain what the edges represent, what the nodes represent, and determine the number of meeting times.	

These five questions required mathematical arguments on subgraphs, graph planarity using Kuratowski's theorem, solving the Chinese Postman Problem, Euler and Hamilton properties, and graph coloring in the context of discussion group scheduling. This instrument was developed by the course lecturer.

Data Analysis

Data analysis was conducted in several stages. Student responses were first described through open coding to identify initial argumentation themes. Next, these responses were aligned with Toulmin's four components: claim, data, warrant, and support based on rigorous operational definitions. Each argumentation component was scored using a graded analytical rubric that assesses completeness, accuracy, and functional quality. Student scores were then categorized based on Table 1. Score categorization.

Table 1. Score categorization for students' argumentation

No.	Category	Score
1.	Very Low	0-20
2.	Low	21-40
3.	Medium	41-60
4.	High	61-80
5.	Very High	81-100

The total argumentation score is obtained by summing the scores from questions 1-5, with each question assigned a different score range based on its difficulty level. Specifically, question 1 ranges from 0-4, question 2 from 0-4, question 3 from 0-4, question 4 from 0-6, and question 5 from 0-7. These varying score ranges reflect the increasing complexity and skill demands required to answer each question. The approach categorized abilities based on Toulmin scores, lecturer considerations, and answer results (Salaam, 2023).

▪ **RESULT AND DISSCUSSION**

This section presents the findings of the research conducted after evaluating students' mathematics argumentation skills in graph theory. The analysis was conducted

to categorize students' levels of argumentation and to explain the features of their reasoning across various problems. The distribution of students in mathematical argumentation is shown in Table 2. The distribution of students in mathematical argumentation is shown in Figure 1.

Table 2. Distribution of students' argumentation levels

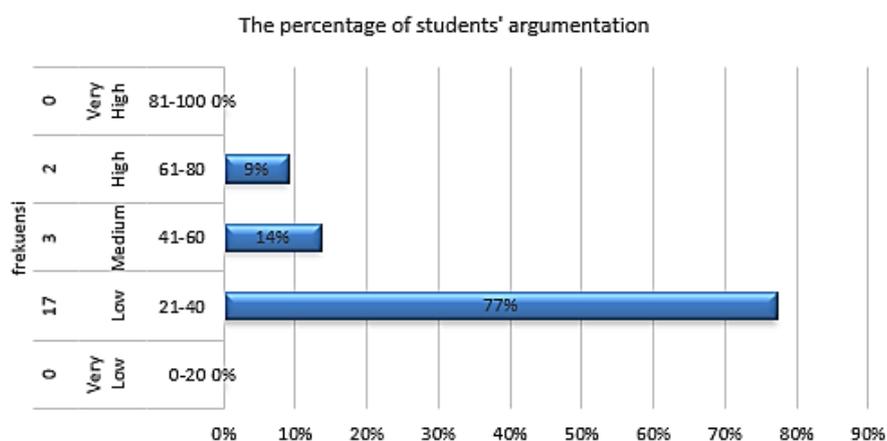| No.	Category	Score	Frequency	Percentage
1.	Very Low	0-20	0	0%
2.	Low	21-40	17	77%
3.	Medium	41-60	3	14%
4.	High	61-80	2	9%
5.	Very High	81-100	0	0

Figure 1. The percentage of students' argumentation across the categories (High, Medium, Low)

Of the 22 students studied, two were High, three were Medium, and the remaining 17 were Low. These findings show that the majority of students exhibited a moderate degree of mathematical argument, indicating that they were capable of reasoning. However, their arguments were usually incomplete or non-theoretical.

Analysis of Students' Answers on the Subgraph Problem

A qualitative analysis of their written responses to each graph theory problem was conducted to understand better how the students constructed their arguments. The initial issue that was studied was the concept of subgraphs. This item was chosen because it would not only help students recognize the relationship between the graphs, but also help them explain why they did so with reference to the definition of a subgraph. The average Toulmin Argumentation Components for the Subgraph Problem are presented in Figure 2.

The first to evaluate students' knowledge of the subgraphies and their uses. The results of the open-code analysis of 22 student answers found differences in ability across all categories. However, high school students (S-1 and S-2) tend to set aside the right claims when complete data is not available, despite inconsistencies in some aspects. For example, S-1 makes accurate claims with relevant data, warranties, and relevant support.
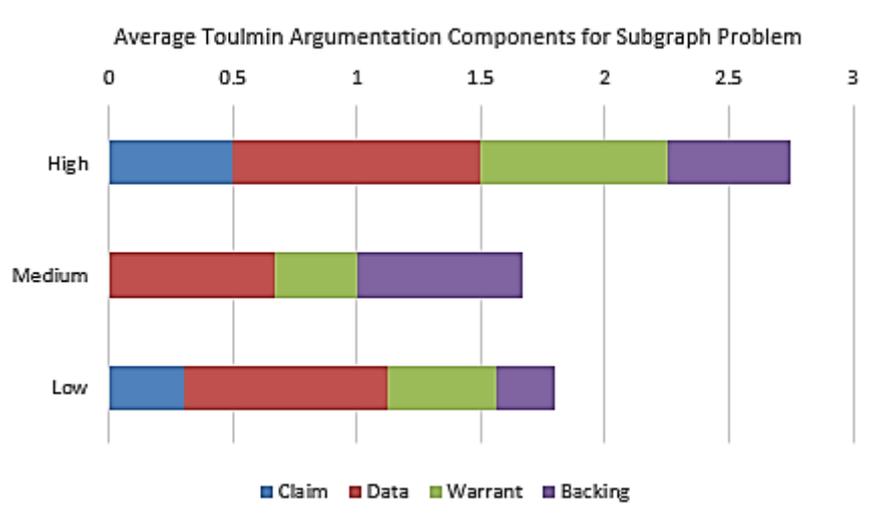
Figure 2. Average toulmin argumentation components for the subgraph problem

The first evaluation examines students' knowledge of the subgraph and its uses, based on the coding analysis of 22 student answers and differences in ability across all categories. However, high-achieving students (S-1 and S-2) tend to produce correct claims supported by complete data, despite several inconsistencies in some aspects. For example, S-1 filed accurate claims with relevant data, guarantees, and support. This shows that even high achievers do not fully understand all components of formal subgraph reasoning, because some of the guarantees they use remain ambiguous and are often based on visualizations rather than concepts. The Medium Category (S-3 to S-5) shows more variety among students. The majority have claims that are partly accurate based on available data, but their explanations are incomplete and theoretically weak. Some students make inaccurate or unsupported statements due to inadequate data. Although some students set out the diagram, they are unable to connect it to a formal subgraph concept. This trend shows that in the middle category, student reasoning is incomplete and tends to be more descriptive than argumentative. Meanwhile, low-level students (S-6 to S-22) struggle to conceptualize subgraph concepts. They seek to provide simple claims and data, but their arguments are unclear and less theoretically rich. Despite their efforts to link visual representation with subgraph concepts, their reasoning is shallow and not mathematical.

This shows that students' ability to construct conceptual mathematical arguments in graph theory is still limited. Specifically, their difficulties often lie in the justification and support stages of argumentation: they may make claims and cite data (perhaps diagrams or visual patterns), but have difficulty (1) articulating a valid justification that connects the data to the claim through theoretical principles, and (2) providing appropriate support based on formal definitions or theorems.

Given these patterns, learning in graph theory must go beyond purely procedural or representational tasks. Instructional design should include activities that explicitly require students to generate justifications and support based on formal theoretical graph definitions and theorems. For example, tasks can ask students not only to draw subgraphs or identify subgraph relationships visually, but also to explain in words, referring to definitions (e.g., definitions of subgraphs, vertex/edge inclusion), comparing and

contrasting examples and non-examples, or justifying why a graph qualifies (or does not qualify) as a subgraph. (Medová et al., 2019).

Daniel & Taneo (2019) research also shows that when graph theory material is presented through teaching materials that emphasize conceptual clarity and relationships between representations, students' ability to construct structural understanding increases significantly. These findings reinforce the research results that students are not sufficiently helped by visual or procedural tasks alone; they need learning support that explicitly links graph representations to formal definitions and the underlying conceptual reasoning.

Analysis of Students' Answers on Kuratowski's Theorem Problem

The next problem focused on Kuratowski's Theorem, which assesses students' understanding of planar graphs and their ability to justify non-planarity using theoretical arguments. This problem required students to recall the formal definition of a planar graph, identify subgraphs, and explain the reasoning supporting their conclusion. The average Toulmin Argumentation Components are presented in Figure 2.
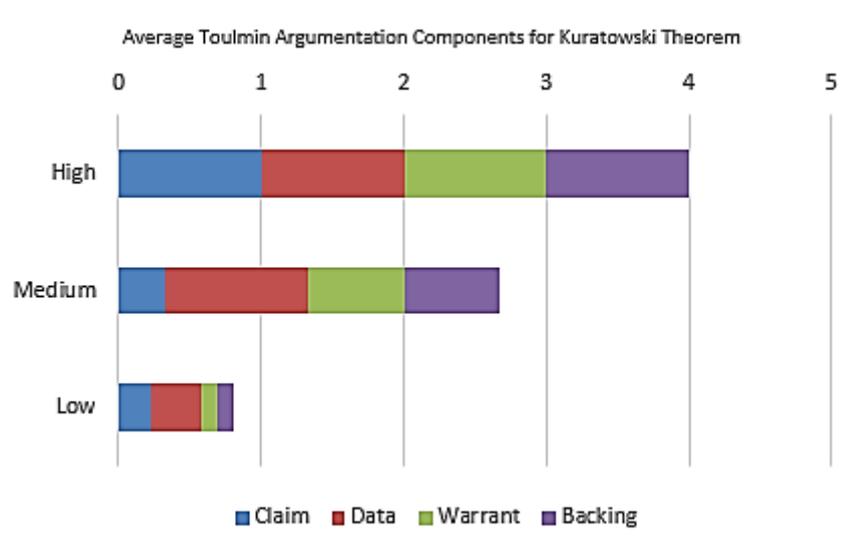
Figure 3. Average toulmin argumentation components for kuratowski's theorem problem

Based on Figure 3, students' performance patterns on Torem-Kuratowski questions revealed a deeper tendency in how they reason about planaritas and how they coordinate visual information with theoretical principles. In the High Category, students consistently achieve the highest scores, suggesting they can provide accurate claims, data, and reasoning. However, the coherence of their answer seems to emerge mainly when the question's structure provides strong clues for applying the theorem. It shows that their success depends more on introducing a familiar pattern, such as the presence of subdivisions, than on a fully internalized conceptual understanding.

This phenomenon explains why some students in the High group show partial knowledge. They can name several theorem conditions but fail to map them onto the given graph, resulting in an incomplete or inaccurate justification. This lack of coherence shows difficulty in aligning the process of visual decomposition of a graph with

Kuratowski's theoretical condition. Similar challenges have been reported in international studies, which found that students often recognize visual features but fail to link them to theoretical invariance (Umberto, 2021).

In the middle group, the pattern of difficulty becomes clearer. Although some students try to attribute their answers to Theorem Kuratowski, the reasoning given was incomplete or inconsistent. This inconsistency mainly arises from three sources: the inability to identify relevant subgraphies, difficulty in connecting visual representations to theoretical conditions (leading to irrelevant justifications), and reliance on memorized definitions, which lead them to cite theorems without explaining their application mechanisms.

As a result, their argument becomes overly procedural, focusing on steps or graphic characteristics without explaining the conceptual reasons for their decision. This pattern is consistent with the findings of Klau et al. (2020), which indicates that students often fail to integrate visual and theoretical representations in advanced math topics. Some students do file true but are unable to provide justification, demonstrating that their conceptual understanding of Kuratowski's Theorem remains fragmented. Memorizing tee or the definition alone does not guarantee the ability to map theories to concrete graphics.

In the low group, answers are very limited, as reflected in the long length of the short components shown in the image. Their argument recognized some graphic characteristics but lacked a theoretical basis. They do not provide assurances, do not refer to the theorem condition, and do not indicate the graph's structural analysis. The pattern is consistent with research showing that students with minimal conceptual understanding tend to rely heavily on visual impressions without accessing a formal definition. Meta-analysis also confirmed that visualization is only effective when accompanied by an explicit hook by formal concepts; without this, visual observation remains shallow and does not support theoretical justification (Schoenherr et al., 2024).

Overall, the pattern in Figure 2 shows that the main source of incoherence in students' arguments is not their inability to remember Kuratowski's Theorem, but their difficulty in integrating three components simultaneously: the visual decomposition of the graph, the formal structure of forbidden subgraphs, and the theoretical justification of planarity. The inability to coordinate these three aspects results in students' arguments being fragmented, incomplete, or misdirected. These findings indicate that students' knowledge is still fragmented and has not been conceptually integrated to support coherent, theory-based mathematical arguments.

Analysis of Students' Answers on the Chinese Postman Problem

The Chinese Postman Problem was the next issue, which evaluated students' ability to apply graph-theoretic concepts to real-world route-optimization problems. Here, students were meant to find the path in a graph with the fewest edges that touches each edge at least once, and to support each step in their line of thinking with a corresponding mathematical argument. Figure 3: Average Toulmin argumentation for the Chinese Postman Problem.

Based on Figure 4 of the Chinese Postman Problems (CPP), students show significant variation in their understanding of the concepts. In the High Category (S-1 and S-2), only one student (S-2) is genuinely able to show a complete understanding of the concepts and algorithms of the CPP. S-2 provides correct claims, complete data, coherent
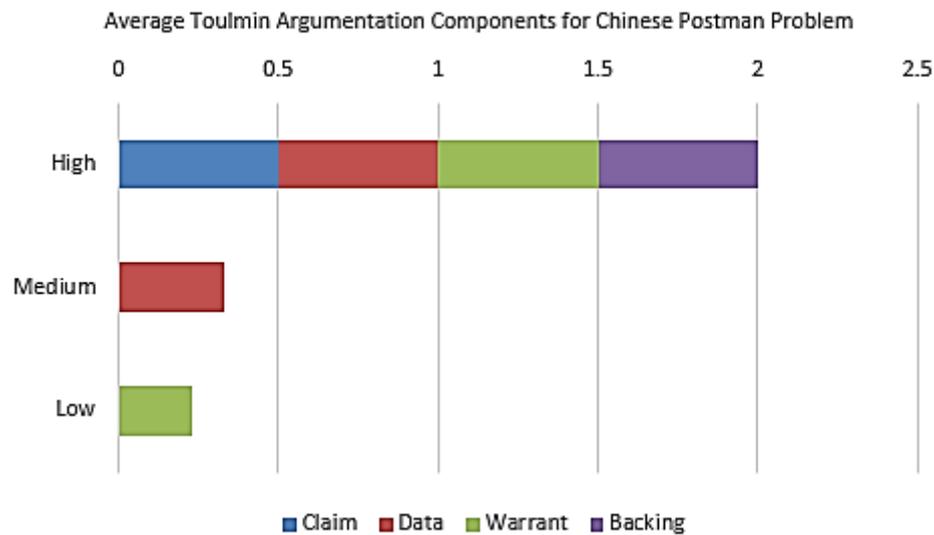
Figure 4. Average toulmin argumentation components for the chinese postman problem

assurance, and theoretical support that resonated with the principles of Graphic Euler. Meanwhile, S-1 still makes conceptual errors, such as misinterpreting an odd-number-matching process or misrepresenting weight, showing that the connection between parity, an odd-number installation, and the formation of the Euler lane remains incomplete. These findings were consistent with research showing that visual representations alone are often insufficient to guide students' understanding of the underlying theoretical structure (Schoenherr & Schukajlow, 2024).

Similar difficulties can also be seen in the Medium category. Most students fail to distinguish between measures: identify odd nodes, conduct a minimum matching, and form the Euler track. Their arguments show wrongful claims or inappropriate data, indicating that they follow procedures mechanically without understanding the logic of minimization within CPP. The phenomenon of procedural-less procedures like this has been widely reported in mathematical representation research, where students tend to take part in the computational steps without understanding its theoretical reasons (Taneo & Daniel, 2025).

Students in the Low category show the most basic difficulties. They made erroneous claims and data; one of them includes an irrelevant warrant, while others do not provide a warrant at all. This pattern indicates a lack of knowledge of problem structure and confusion between CPP and other track problems. This finding is in line with studies showing that students with minimal conceptual understanding tend to depend on graph visual features without connecting them with formal definitions or theoretical invariants (Muzangwa & Ogbonnaya, 2022). Overall, these results show that students' conceptual understanding of the traversal graph, ditches, and the formation of the Euler circuit remains low.

Analysis of Students' Answers on Euler and Hamilton Circuits

The subsequent problem focused on Euler-Hamilton Circuits, designed to evaluate students' understanding of traversability and cyclic paths in graphs. Students were

required to distinguish between Eulerian and Hamiltonian properties, identify corresponding circuits, and provide logical justification for their conclusions. Average Toulmin Argumentation Components for Euler & Hamilton Problem on Figure 5.
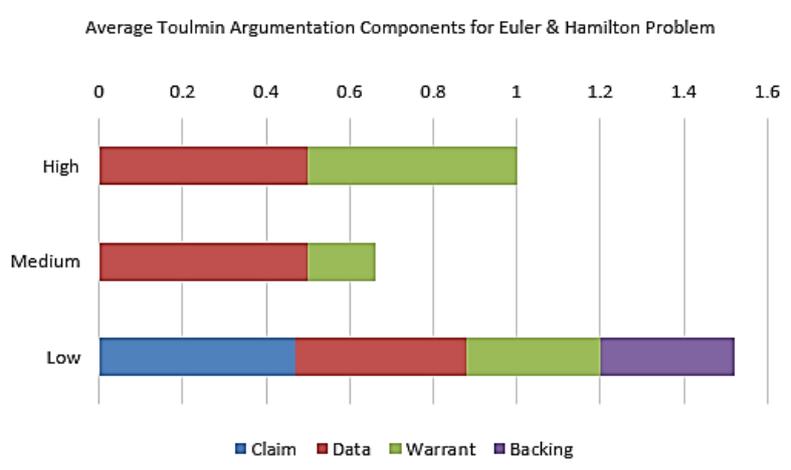

Figure 5. Average toulmin argumentation components for euler & hamilton problem

Based on Figure 5, the student performance patterns for Euler and Hamilton showed that their ability to link graph representations with theoretical concepts remains unstable across categories. In the High (S-1 and S-2) categories, scores indicate that, while they can identify basic elements such as the degree of a node or the possibility of a path, their reasoning is incomplete and is not always supported by the correct theory. Some students do try to present claims and solution steps. However, the data used is often incorrect, the justification is only partial, and the link between visual findings and the formal definition of the Euler circuit or Hamilton's Hamiltonian is not strong. The phenomenon indicates that their conceptual understanding of circuit requirements, such as the fact that all nodes must have even degree for the Euler circuit or that each node must be visited exactly once for the Hamilton circuit, has not been fully internalized.

In the Medium (S-3 to S-5 category), student performance is more stable. Some students provide correct claims with in-person data and start showing relevant justifications. This shows that they begin to link visual graphs to theoretical concepts, though the connections remain inconsistent and often incomplete. Conversely, some students in the group still show an incorrect relationship among claims, data, and justification, indicating that the basic concepts of the Euler circuit and Hamilton have not been fully developed. Even though most of them may write or draw circuit representations, the justifications they provide are not always aligned with formal principles, suggesting that coordination among the procedural, representational, and conceptual aspects is still in the developmental stage.

The Low category, when viewed on average per component, still showed lower-than-optimal argumentation quality. Although they filled out all component claims down to the backing, their quality was inconsistent, their data was less relevant, and their justifications did not align with the principles of graphical theory. This suggests that students in the Low group still struggled to connect visual representations of Eulerian and Hamiltonian circuits with the corresponding theoretical requirements. The study's

findings indicate that most students can only recognize the visual representations of the Euler and Hamilton tracks but are unable to link them to the appropriate theoretical requirements. These results are in line with González et al. (2021) which finds that student reasoning in graph theory is generally at the entry level of Hiele van, namely, the visual phase and basic analysis. At this level, students tend to focus on the visible characteristics of a graph but have not been able to associate it with formal definitions or perform legal deductions. González et al. (2021) also confirms that this level-level characterization is important as the basis for developing a reasoning evaluation instrument and calls for an empirical study to map students' thinking skills in graph theory. Thus, this study also strengthens and complements the theoretical study by analyzing the students' arguments on the trajectories of Euler and Hamilton.

Analysis of Students' Answers on Graph Coloring

The last task, Graph Coloring, focused on testing students' ability to reason about combinations to color the vertices according to given restrictions. Students were to identify the chromatic number of a given graph, justify the coloring choices in their answers, and justify the minimality of their answers. Figure 6 Average Toulmin Argumentation Components for Graph Coloring Problem
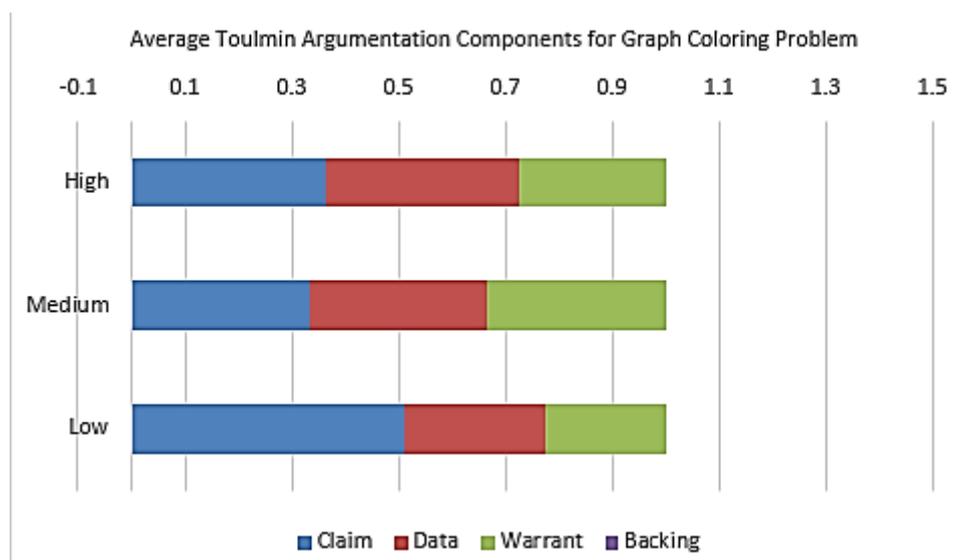
Figure 6. Average toulmin argumentation components for graph coloring problem

Based on Figure 6, High Category Students (S-1 and S-2) tend to provide correct, relatively complete solutions to graph-coloring problems in the scheduling context. Their answers contain appropriate claims, relevant information, and logical reasoning, especially in explaining that the adjacent nodes must be given different colors. However, students in this category still score poorly because they have not fully connected their steps and end answers to the underlying graph-drawing theory. In other words, they understand procedures and can reach the correct conclusion, but their conceptual knowledge and theoretical support remain limited.

Medium Category (S-3 to S-5) students can present correct claims and identify nodes and their associated sides with scheduling issues. However, the support

data they provide is minimal and incomplete. Most of them do not explain the reasons behind the selected color or how their answers are associated with principles or theorems in graph theory. Therefore, their responses tend to present clear conclusions but not conceptual ones. Although they have a basic procedural understanding, the mathematical arguments they draft have been incoherent in their claims, data, and theoretical foundations.

Meanwhile, Low Category (S-6 to S-22) students show an increase at the initial level. Some can provide accurate claims, but only with very limited information, while others make claims without any supporting explanation. Their arguments lack assurances or theoretical support, as their understanding of the fundamental concept of graph coloring remains very limited. In general, they might mention correct measures or terminology, but are not capable of developing complete arguments or connecting their answers to a theoretical basis. Overall, these findings suggest that most students can perform graph-dyeing procedures, but their theoretical understanding still needs improvement.

The results of the study indicate that students' mathematical argumentation abilities vary by ability level. High students can construct more complete arguments, with correct statements, relevant facts, and strong foundations, but they do not provide theoretical support. Medium students make partially correct claims and incomplete reasoning, and they do not provide theoretical support. Low students make simple or false claims without specific data or theory to support them.

These results agree with those of Demircioglu et al. (2023), who noted that all students can engage in argumentation. However, the quality of their arguments is closely tied to their ability to reason. That is, reasoning capacity is what makes the difference between students who can merely assert simple statements and those who can formulate thorough and theoretically based arguments. This trend was also evident in this research: the better students' argumentation skills, the better the organization of their arguments, in terms of the clarity of claims, data sufficiency, warrant strength, and the relevance of theoretical support. These observations align with Uzun's (2024) findings on the process of mathematical argumentation following alterations to the structure of the national examination that required higher-order reasoning skills. In the research, students who learned in a setting that promoted group debate and argument developed stronger mathematical reasoning skills. The analysis found that argument structure varied across classes, and those with more discussion and interaction produced higher-quality arguments. This confirms the findings of the current study, indicating that mathematical argumentation ability is highly dependent on reasoning level and on learning situations in which learners can argue, defend, and substantiate their statements in a logical manner.

Moreover, the results of this research are supported by the theoretical premises provided by Hurrell (2021), which state that the ability to engage in mathematical reasoning depends on the depth and integration of students' conceptual and procedural knowledge. Hurrell states that the better students have connected conceptual knowledge, the more they can create logical explanations and justifications. Contrarily, learners with disjointed or shallow learning are likely to support half-cooked statements without sufficient arguments. The theoretical approach would help explain why the students who achieved high scores in this study produced more complete and logically organized arguments. On the contrary, poor students could not supply information, justification, or

theoretical support. As Mukuka & Tatira (2025) found, when making arguments or proofs, learners tend to rely on procedural familiarity rather than conceptual understanding. Their participants often made unsupported claims with insufficient evidence, thus having loopholes in logic. These recurring challenges align with the patterns of argumentation revealed in this research, especially among students in the medium and low categories, who favored producing incomplete or poorly motivated arguments. In this way, their results provide powerful empirical evidence that discrepancies in the quality of argumentation can be deeply grounded in differences in students' reasoning capacities.

The results of the present study are highly supported by these findings of the Berkle et al. (2023) study. In their research, they established that university students have significant difficulty determining the elements of an argument, especially the claim and the warrant, which are vital to forming a consistent argument. On the same lines, the participants' work tended to generate incomplete arguments due to the use of intuitive thinking (Type I) rather than analytic thinking (Type II), as observed in the present study. This led to their inclination to accept fallacies, to ignore logical links, and to fail to substantiate their arguments. These conclusions are largely consistent with current research showing that students with low reasoning skills have a lower capacity to construct arguments based on logic and often simply put forward more basic statements with no supporting data, warrants, or theoretical support. In this way, the study by Berkle et al. contributes to the view that argumentation is a direct response to students' reasoning abilities. Additional evidence is provided by Muhlen et al. (2019), who discovered that the reasoning capacity of students to recognize the structure of an argument is highly contingent on their reasoning capacity: students with greater reasoning skills were more successful in recognizing claim-data-warrant relations. They received more benefits from structured argumentation training.

This trend is supported by evidence from studies carried out in Indonesia. Arifin et al. (2023) found that Indonesian university students had a high rate of failing to meet the structural elements of argumentation, e.g., providing warrants and supporting evidence, as well as lacking the conceptual base necessary to create valid mathematical demonstrations. Such findings are similar to those of the current study, in which low-ability students were not only unable to provide sufficient data or warrants but also had conceptual gaps that hindered the connection between evidence and claims. On the other hand, more talented students, like the few successful students in this research, were more deeply grounded in the concept, thus able to create more coherent and logically organized arguments. Further evidence of this research's results is that, according to Lubis and Lubis (2024), High-ability students were able to make claims, provide evidence, and reason, while medium-ability students could make claims and use data only. The low-ability students regularly made false claims and failed to support their assertions with arguments or counterarguments. In that way, the Indonesian situation is another reinforcement of the meaning that the inabilities of the mathematical argumentation are not only caused by the lack of the ability to reason, but also by the lack of the necessary development of the conceptual knowledge, which is needed to create the full, justified, and theoretically based arguments (Arifin et al., 2023; Lubis & Lubis, 2024).

Although previous studies have shown that weaknesses in mathematical reasoning generally stem from low reasoning ability and poor conceptual mastery (Demircioglu et

al., 2023; Uzun, 2024; Hurrell, 2021; Mukuka & Tatira, 2025), this study provides additional theoretical contributions that have not been widely discussed in previous research. Specifically, this study's results expand on previous findings by showing that the characteristics of argumentation difficulties in graph theory differ from those in other mathematical domains, such as algebra or calculus. In graph theory, students are not only required to understand formal definitions such as subgraphs, sets of vertices, and adjacency relations, but also to connect the visual representation of graphs with abstract theoretical justifications. This study found that most students could recognize the visual form of subgraphs but failed to connect these observations to formal definitions or relevant graph-theoretic principles. This pattern is more prominent than findings in the domains of algebra or calculus, where argumentation errors are more often related to procedures or symbolic manipulation.

All these studies support the major finding of the current study: the quality of mathematical argumentation is primarily defined by students' reasoning ability. Students with higher reasoning abilities make more explicit statements, use more relevant data, provide stronger warrants, and offer more appropriate theoretical support. By contrast, individuals with weaker reasoning abilities find it difficult to present logical relationships, select applicable evidence, or support their claims. Therefore, the trend in the given study is not an exception: argumentation competence does not only concern the ability to generate answers. Even then, it demonstrates deep discrepancies in thinking, in the perception of concepts, and in the ability to synthesize the elements of a logical argument.

The main limitation of this study is that it relies on students' midterm exam answers as its primary data source. These answers do not fully reflect the breadth of students' thinking processes, as time constraints and the high-stakes nature of exams can limit the depth and clarity of their arguments. In addition, the relatively small sample size limits the generalization of findings beyond the specific context studied. These limitations need to be considered because they may affect the patterns of mathematical argumentation observed and reduce the extent to which conclusions can be applied to a broader student population.

▪ **CONCLUSION**

This study concludes that students' mathematical argumentation abilities in graph theory vary significantly across ability levels and are shaped not only by differences in reasoning and procedural skills but, more fundamentally, by their conceptual understanding of graph theory. High-ability students presented accurate claims, relevant data, and logical warrants; however, their frequent omission of theoretical backing suggests that their conceptual understanding, particularly of formal definitions, properties, and theorems, remains only partially integrated into their reasoning. Medium-ability students displayed even more apparent conceptual limitations. Although they could state claims and identify data points, their difficulty in articulating warrants or connecting steps to underlying graph-theoretic principles shows that their understanding remained procedural and lacked conceptual grounding. Low-ability students exhibited the most substantial conceptual gaps, producing incorrect claims and disconnected data, revealing limited mastery of foundational concepts such as subgraph relationships, planarity, vertex parity, Eulerian conditions, and chromatic constraints. Research findings on various topics in graph theory show that students' arguments reflect both conceptual

difficulties and several strengths that can be used as a basis for further development. Although many students can identify important components of a graph and recognize relevant visual patterns, they sometimes have difficulty connecting these representations with formal definitions or theoretical concepts. High-level students demonstrate that the basis for more robust conceptual integration is already present by showing procedural accuracy and articulating partial reasons. Although these efforts are sometimes inconsistent or inadequate, students with moderate and low abilities show early forms of conceptual awareness, such as observing structural features or attempting to apply definitions. This initial competence highlights that, with targeted support, students have meaningful potential to develop more coherent and theory-based mathematical arguments.

These findings confirm that students' difficulties in constructing complete mathematical arguments are not only due to limited logical skills, but also to a lack of conceptual mastery and an inability to integrate concepts with procedures. Effective argumentation in graph theory requires an understanding of how definitions, theorems, and conceptual relationships support each step of reasoning, a connection that most students do not yet fully understand. Therefore, instructional designs in discrete mathematics should integrate concept-focused activities in graph theory, such as unpacking definitions, examining theorem conditions, and linking procedures to underlying principles, to strengthen students' ability to justify their arguments theoretically, guiding students to interpret definitions meaningfully, connect procedural steps with theoretical principles, and articulate concept-based justifications. Strengthening conceptual understanding alongside reasoning and procedures will enable students to construct arguments that are not only logical but also theoretically grounded and mathematically robust. However, these implications must be considered with the limitations of this study. The use of test-based written responses may not fully capture the depth of students' reasoning, as time constraints may limit how fully students express their arguments. In addition, the relatively small, context-specific sample limits the generalizability of the findings. Future research should directly test students' conceptual understanding of graph theory through interviews or task-based observations. Studies with larger and more diverse samples are also needed to improve generalizability. Furthermore, instructional design research could explore which teaching approaches best support students in integrating concepts with procedures when constructing mathematical arguments.

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- **APPENDIX**

Table 3. Coding of students' answers on the subgraph problem

Argumentations Category	Code of participant	Score	Coding of Mathematical Argumentations
High	S-1	4	Correct claim; complete data; appropriate warrant; includes backing

	Code	Score	Coding
	S-2	1	Incorrect claim; data available; unclear warrant; no backing
Medium	S-3	2	Incorrect claim; data available; unclear warrant; includes backing (diagram)
	S-4	2	Incorrect claim; data available; unclear warrant; includes backing (diagram)
	S-5	0	Did not answer correctly
	S-6	1	Incorrect claim; data available; unclear warrant; no backing
	S-7	2	Incorrect claim; data available; unclear warrant; includes backing (e.g., diagram)
	S-8	2	Incorrect claim; data available; unclear warrant; includes backing (e.g., diagram)
	S-9	2	Partially Correct claim; data available; unclear warrant; no backing
	S-10	2	Partially Correct claim; data available; unclear warrant; no backing
	S-11	2	Partially Correct claim; data available; unclear warrant; no backing
	S-12	1	Incorrect claim; data available; unclear warrant; no backing
	S-13	2	Partially Correct claim; data available; unclear warrant; no backing
Low	S-14	1	Incorrect claim; data available; unclear warrant; no backing
	S-15	3	Partially correct claim; partially correct data; incomplete warrant; no backing
	S-16	1	Incorrect claim; data available; unclear warrant; no backing
	S-17	2	Incorrect claim; data available; unclear warrant; includes backing
	S-18	2	Partially Correct claim; data available; unclear warrant; no backing
	S-19	0	Did not answer correctly
	S-20	0	Did not answer correctly
	S-21	2	Partially correct claim; data available; unclear warrant; no backing
	S-22	3	Partially correct claim; partially correct data; incomplete warrant; includes backing

Table 4. Coding of students' answers on kuratowski's theorem problem

Argumentations Category	Code of participant	Score	Coding of Mathematical Argumentations
High	S-1	4	Correct claim; accurate data; application of Kuratowski's Theorem
	S-2	4	Correct claim; accurate data; application of Kuratowski's Theorem
Medium	S-3	4	Correct claim; accurate data; application of Kuratowski's Theorem

	S-4	1	Incorrect claim; some data provided; no warrant; no backing
	S-5	3	Incorrect claim; accurate data; application of Kuratowski's Theorem
	S-6	2	Incorrect claim; data available; unclear warrant; no backing
	S-7	0	Incorrect claim; incorrect data; unclear warrant; no backing
	S-8	0	Incorrect claim; incorrect data; unclear warrant; no backing
	S-9	0	Incorrect claim; incorrect data; unclear warrant; no backing
	S-10	1	Correct claim; some data provided; no warrant; no backing
	S-11	3	Incorrect claim; accurate data; application of Kuratowski's Theorem
	S-12	2	Incorrect claim; data available; unclear warrant; no backing
	S-13	0	Partially Correct claim; data available; unclear warrant; no backing
Low	S-14	3	Incorrect claim; accurate data; application of Kuratowski's Theorem
	S-15	0	Incorrect claim; incorrect data; unclear warrant; no backing
	S-16	2	Incorrect claim; data available; unclear warrant; no backing
	S-17	0	Incorrect claim; incorrect data; unclear warrant; no backing
	S-18	2	Partially Correct claim; data available; unclear warrant; no backing
	S-19	1	Correct claim; some data provided; no warrant; no backing
	S-20	0	Did not answer correctly
	S-21	1	Correct claim; some data provided; no warrant; no backing
	S-22	1	Correct claim; some data provided; no warrant; no backing

Table 5. Coding of students' answers on the chinese postman problem

Argumentations Category	Code of participant	Score	Coding of Mathematical Argumentations
	S-1	0	Did not answer correctly
High	S-2	4	Correct claim; complete data (total weight + matching of odd vertices); logical warrant; includes backing (definition of Chinese Postman Problem)
	S-3	0	Did not answer correctly
Medium	S-4	0	Incorrect claim; some data provided; no warrant; no backing

	Code of participant	Score	Coding of Mathematical Argumentations
	S-5	1	Incorrect but nearly correct claim; data provided but inaccurate; no warrant; no backing
	S-6	1	Incorrect but nearly correct claim; data provided but inaccurate; no warrant; no backing
	S-7	0	Incorrect claim; incorrect data provided; no warrant; no backing
	S-8	0	Incorrect claim; incorrect data provided; includes warrant; no backing
	S-9	0	Did not answer correctly
	S-10	0	Incorrect claim; incorrect data provided; includes warrant; no backing
	S-11	0	Incorrect claim; incorrect data provided; no warrant; no backing
	S-12	0	Did not answer correctly
	S-13	0	Incorrect claim; incorrect data provided; no warrant; no backing
Low	S-14	0	Incorrect claim; incorrect data provided; includes warrant; no backing
	S-15	0	Irrelevant answer
	S-16	0	Irrelevant answer
	S-17	1	Incorrect but nearly correct claim; data provided but inaccurate; no warrant; no backing
	S-18	0	Irrelevant answer
	S-19	1	Incorrect but nearly correct claim; data provided but inaccurate; no warrant; no backing
	S-20	0	Did not answer correctly
	S-21	0	Incorrect claim; incorrect data provided; no warrant; no backing
	S-22	0	Incorrect claim; incorrect data provided; includes warrant; no backing

Table 6. Coding of students' answers on euler and hamilton circuits

Argumentations Category	Code of participant	Score	Coding of Mathematical Argumentations
High	S-1	3	Incorrect claim, but partially correct data; incomplete warrant; no backing
	S-2	3	Incorrect claim, but partially correct data; incomplete warrant; no backing
	S-3	2	Claim present but incorrect; partially correct data; no warrant; no backing
Medium	S-4	3	Incorrect claim, but partially correct data; incomplete warrant; no backing
	S-5	2	Claim present but incorrect; partially correct data; no warrant; no backing
Low	S-6	2	Claim present but incorrect; partially correct data; no warrant; no backing
	S-7	4	Correct claim; partially correct data; warrant present; includes backing

	S-8	3	Correct claim; partially correct data; warrant present; partial backing
	S-9	4	Correct claim; partially correct data; warrant present; includes backing
	S-10	2	Incorrect claim; partially correct data; warrant present; partial backing
	S-11	3	Correct claim; partially correct data; warrant present; partial backing
	S-12	4	Correct claim; partially correct data; warrant present; includes backing
	S-13	5	Correct claim; partially correct data; warrant present; includes backing
	S-14	2	Incorrect claim; incorrect data provided; includes warrant; no backing
	S-15	3	Incorrect claim, but partially correct data; incomplete warrant; no backing
	S-16	2	Claim present but incorrect; partially correct data; no warrant; no backing
	S-17	2	Incorrect but nearly correct claim; data provided but inaccurate; no warrant; no backing
	S-18	1	Incorrect claim; partial data provided; no warrant; no backing
	S-19	2	Incorrect but nearly correct claim; data provided but inaccurate; no warrant; no backing
	S-20	4	Correct claim; partially correct data; warrant present; includes backing
	S-21	1	Incorrect claim; partial data provided; no warrant; no backing
	S-22	1	Incorrect claim; partial data provided; no warrant; no backing

Table 7. Coding of students' answers on graph coloring

Argumentations Category	Code of participant	Score	Coding of Mathematical Argumentations
High	S-1	5	Correct claim; complete data; logical warrant (mentions graph coloring, adjacent vertices must have different colors); no backing
	S-2	4	Correct claim; fairly complete data; partial warrant; no backing
Medium	S-3	5	Correct claim; complete data; logical warrant (mentions graph coloring, adjacent vertices must have different colors); no backing
	S-4	5	Correct claim; complete data; logical warrant (mentions graph coloring, adjacent vertices must have different colors); no backing
	S-5	5	Correct claim; complete data; logical warrant (mentions graph coloring, adjacent vertices must have different colors); no backing

	S-6	4	Correct claim; fairly complete data; partial warrant; no backing
	S-7	3	Correct claim; partial data (vertices or edges mentioned but incomplete); no warrant; no backing
	S-8	4	Correct claim; fairly complete data; partial warrant; no backing
	S-9	3	Correct claim; partial data (vertices or edges mentioned but incomplete); no warrant; no backing
	S-10	4	Correct claim; fairly complete data; partial warrant; no backing
	S-11	1	Correct claim; no data; no warrant; no backing
	S-12	2	Correct claim; partial data; no warrant; no backing
	S-13	2	Correct claim; partial data; no warrant; no backing
	S-14	2	Correct claim; partial data; no warrant; no backing
Low	S-15	2	Correct claim; partial data; no warrant; no backing
	S-16	3	Correct claim; partial data (vertices or edges mentioned but incomplete); no warrant; no backing
	S-17	3	Correct claim; partial data (vertices or edges mentioned but incomplete); no warrant; no backing
	S-18	3	Correct claim; partial data (vertices or edges mentioned but incomplete); no warrant; no backing
	S-19	3	Correct claim; partial data (vertices or edges mentioned but incomplete); no warrant; no backing
	S-20	3	Correct claim; partial data (vertices or edges mentioned but incomplete); no warrant; no backing
	S-21	2	Correct claim; partial data; no warrant; no backing
	S-22	1	Correct claim; no data; no warrant; no backing